k-defects as compactons

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Corrigendum

k-defects as compactons

Adam C, Sanchez-Guillen J and Wereszczynski A 2007 *J. Phys. A: Math. Theor.* **40** 13625 (arXiv:0705.3554)

In Adam *et al* (2007 *J. Phys. A: Math. Theor.* **40** 13625), a compact soliton solution has been constructed and its stability under linear fluctuations has been proved. The stability proof in section 4.4 of this paper is, however, incorrect. In this corrigendum we provide the correct proof.

In [1] an explicit solution of a compact topological soliton has been given, and in section 4.4 of this paper the stability of the compact soliton under linear fluctuations has been proved. The proof of stability in this paper is, however, erroneous. The statement itself is correct (i.e., the compact soliton is stable under linear fluctuations), and the correct proof is provided in the following. As the error affects most of section 4.4, we prefer to simply rewrite this section, so the text below just is the new, correct version of section 4.4 of [1].

In this corrected version of subsection 4.4 of [1] we shall demonstrate the linear stability of the compactons of section 4.3 of this reference. Here we closely follow the stability analysis of [2]. We introduce general fluctuations around a static (compacton) solution, $\xi(x,t) = \xi(x) + \eta(x,t)$ (here, $\xi(x)$ is the compacton solution and $\eta(x,t)$ is the fluctuation field), and insert this expression into the action of a general Lagrangian $L(v,\xi)$ (remember $v \equiv (1/2)\xi^\mu\xi_\mu$). The part of the action quadratic in the fluctuation η, which is relevant for the stability analysis, is

$$S^{(2)} = \int \mathrm{d}^2x \left(\frac{1}{2}L_v\eta^\mu\eta_\mu + L_{vv}\frac{1}{2}(\xi^\mu\eta_\mu)^2 + L_{\xi\xi}\frac{1}{2}\eta^2 + L_{\xi v}\eta\xi^\mu\eta_\mu\right) \tag{1}$$

or, after using the identity

$$2L_{\xi v}\eta\xi^\mu\eta_\mu = \partial_\mu(L_{\xi v}\eta^2\xi^\mu) - \eta^2\partial_\mu(L_{\xi v}\xi^\mu), \tag{2}$$

$$S^{(2)} = \frac{1}{2}\int \mathrm{d}^2x(L_v\eta^\mu\eta_\mu + L_{vv}(\xi^\mu\eta_\mu)^2 + L_{\xi\xi}\eta^2 - \partial_\mu(L_{\xi v}\xi^\mu)\eta^2). \tag{3}$$

The linear equation for the fluctuation field following from this action is

$$\partial_\mu(L_v\eta^\mu + L_{vv}\xi^\mu\xi_\alpha\eta^\alpha) - [L_{\xi\xi} - \partial_\mu(L_{\xi v}\xi^\mu)]\eta = 0. \tag{4}$$

Now we take into account that ξ is static, and we replace v by its static limit $w \equiv -(1/2)\xi_x^2$. Further, we assume that

$$\eta(x,t) = \cos(\omega t)\eta(x). \tag{5}$$

The resulting linear ODE for $\eta(x)$ is

$$-\partial_x[(L_w + 2L_{ww}w)\eta_x] - [L_{\xi\xi} + \partial_x(L_{\xi w}\xi_x)]\eta = \omega^2 L_w\eta. \tag{6}$$

For the specific class of Lagrangians $L = F(v) - U(\xi)$ this simplifies to

$$-\partial_x[(F_w + 2F_{ww}w)\eta_x] + U_{\xi\xi}\eta = \omega^2 F_w \eta. \tag{7}$$

Next, we specialize to the Lagrangian of section 4.3 of [1]

$$F = 4\tilde{M}^2|w|w, \qquad U = 3\lambda^2(\xi^2 - a^2)^2, \tag{8}$$

and arrive at the equation

$$-12\tilde{M}^2\partial_x\left(\xi_x^2\eta_x\right) + 12\lambda^2(3\xi^2 - a^2)\eta = 4\tilde{M}^2\omega^2\xi_x^2\eta. \tag{9}$$

This expression must now be evaluated for the compacton solution $\xi(x)$ of section 4.3 of [1]. In the outer region of the compacton, i.e., in the region $|x| > \frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}}$ where $\xi = \pm a =$ const., obviously only the solution $\eta = 0$ is possible. As we want η to be continuous at the boundary of the compacton, a general $\eta(x)$ should go to zero at the compacton boundaries. The corresponding space of functions may be divided into an even and an odd subspace under the reflection $x \to -x$, and basis functions for the two subspaces are

$$\eta_n(x) = \begin{cases} 0 & x \leqslant -\frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \\ \cos(2n+1)\sqrt{\frac{\lambda}{\tilde{M}}}x & -\frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \leqslant x \leqslant \frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \\ 0 & x \geqslant \frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \end{cases} \tag{10}$$

for the even subspace (here $n = 0, \ldots, \infty$) and

$$\zeta_m(x) = \begin{cases} 0 & x \leqslant -\frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \\ \sin 2m\sqrt{\frac{\lambda}{\tilde{M}}}x & -\frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \leqslant x \leqslant \frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \\ 0 & x \geqslant \frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \end{cases} \tag{11}$$

for the odd subspace (here $m = 1, \ldots, \infty$). The restriction on this class of functions will be important in the stability analysis below. Observe that the first derivative of η is not continuous at the boundary. This is consistent with the fact that the compacton itself is continuous together with its first derivative. Also, equation (9) is well defined everywhere, because η_x is always multiplied by zero at the points of discontinuity.

For linear stability, the eigenvalue ω^2 on the rhs of equation (9) has to be positive semi-definite, $\omega^2 \geqslant 0$. For this to hold, the linear differential operator acting on η on the lhs of equation (9) should be a positive semi-definite operator on the space of functions (10) and (11). In order to demonstrate this, we rewrite equation (9) as

$$\tilde{H}\eta = 4\tilde{M}^2\omega^2\xi_x^2\eta, \tag{12}$$

where

$$\tilde{H} = -12a^2\tilde{M}\lambda\cos^2\sqrt{\frac{\lambda}{\tilde{M}}}x\,\partial_x^2 + 24a^2\lambda^{\frac{3}{2}}\tilde{M}^{\frac{1}{2}}\sin\sqrt{\frac{\lambda}{\tilde{M}}}x\cos\sqrt{\frac{\lambda}{\tilde{M}}}x\,\partial_x + 12\lambda^2a^2\left(3\sin^2\sqrt{\frac{\lambda}{\tilde{M}}}x - 1\right). \tag{13}$$

It is useful to introduce the new coordinate $y = \sqrt{\frac{\lambda}{\tilde{M}}}x$ and to rewrite

$$\tilde{H} = 12a^2\lambda^2 H \tag{14}$$

with

$$H = -\cos^2 y\partial_y^2 + 2\sin y\cos y\partial_y + 3\sin^2 y - 1. \tag{15}$$

We now want to demonstrate that the operator H is positive semi-definite on the space of functions which are zero for $|y| \geqslant \frac{\pi}{2}$ and continuous at the compacton boundaries $y = \pm\frac{\pi}{2}$. This space may be divided into an even and an odd subspace under the reflection $y \to -y$, and these two subspaces may be treated separately, because the operator H is even and does not mix the two subspaces. A basis for the even subspace is (here $n = 0, \ldots, \infty$)

$$\eta_n(y) = \begin{cases} 0 & y \leqslant -\dfrac{\pi}{2} \\ \cos(2n+1)y & -\dfrac{\pi}{2} \leqslant y \leqslant \dfrac{\pi}{2} \\ 0 & y \geqslant \dfrac{\pi}{2}, \end{cases} \tag{16}$$

whereas a basis for the odd subspace is (here $m = 1, \ldots, \infty$)

$$\zeta_m(y) = \begin{cases} 0 & y \leqslant -\dfrac{\pi}{2} \\ \sin 2my & -\dfrac{\pi}{2} \leqslant y \leqslant \dfrac{\pi}{2} \\ 0 & y \geqslant \dfrac{\pi}{2}. \end{cases} \tag{17}$$

We remark that the (correct) basis functions displayed here differ from the (incorrect) ones in [1], see equations (61) and (66) of [1], which is the first error in this reference.

Next, we want to prove the positive semi-definiteness of H on the two subspaces. For the even subspace we find

$$\begin{aligned}\cos(2m+1)yH\cos(2n+1)y &= \left(n^2+n+\tfrac{1}{2}\right)[\cos 2(m-n)y + \cos(2(m+n+1)y] \\ &+ \tfrac{1}{2}(n^2-1)[\cos 2(m-n+1)y + \cos 2(m+n)y] \\ &+ \tfrac{1}{2}(n+2n)[\cos 2(m-n-1)y + \cos 2(m+n+2)y]\end{aligned} \tag{18}$$

and, therefore,

$$\begin{aligned}\langle m|H|n\rangle &\equiv \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}y\cos(2m+1)yH\cos(2n+1)y \\ &= \pi\left[\left(n^2+n+\frac{1}{2}\right)(\delta_{m,n} - \delta_{m,0}\delta_{n,0}) + \frac{1}{2}(n^2-1)\delta_{m,n-1} + \frac{1}{2}(n^2+2n)\delta_{m,n+1}\right].\end{aligned} \tag{19}$$

It obviously holds that $\langle n|H|n\rangle \geqslant 0$ for all n, but this is only a necessary and not a sufficient condition for the positive semi-definiteness of H. This is the second error of [1].

We have to demonstrate positive semi-definiteness for a general vector

$$|v\rangle = \sum_{n=0}^{\infty} c_n\cos(2n+1)y, \tag{20}$$

where, however, we will restrict to normalizable vectors v. A normalizable vector may always be approximated to arbitrary precision by a vector

$$|v\rangle = \sum_{n=0}^{N} c_n \cos(2n+1)y \tag{21}$$

for sufficiently large but finite N; therefore, we will restrict to this case in the following.

Before continuing, we remark that the basis function $\cos y$ for $n = 0$ is a zero mode of the operator H, which is related to the translational invariance of the compactons, see [2] for a more-detailed discussion. Therefore, all matrix elements with $m = 0$ or $n = 0$ are zero, and we may assume $c_0 = 0$ without loss of generality. Taking this fact into account, we find

$$\begin{aligned}
\langle v|H|v\rangle &= \sum_{m,n=1}^{N} c_n \bar{c}_m \langle m|H|n\rangle \\
&= \pi \sum_{n=1}^{N} \left[c_n \bar{c}_n \left(n^2 + n + \frac{1}{2} \right) + c_n \bar{c}_{n-1} \frac{1}{2}(n^2 - 1) + c_n \bar{c}_{n+1} \frac{1}{2}(n^2 + 2n) \right] \\
&\geqslant \pi \sum_{n=1}^{N} \left[c_n \bar{c}_n \left(n^2 + n + \frac{1}{2} \right) - |c_n||\bar{c}_{n-1}| \frac{1}{2}(n^2 - 1) - |c_n||\bar{c}_{n+1}| \frac{1}{2}(n^2 + 2n) \right] \\
&= \pi \sum_{n=1}^{N} \left[|c_n|^2 \left(n^2 + n + \frac{1}{2} \right) - |c_n||c_{n-1}|(n^2 - 1) \right],
\end{aligned} \tag{22}$$

where $c_{N+1} \equiv 0$ by assumption. We now want to prove that the above expression is positive semi-definite. The positive semi-definiteness of this expression is implied by the inequality

$$\sum_{n=1}^{N} [|c_n|^2 - |c_n||c_{n-1}|] \geqslant 0 \tag{23}$$

because of the inequality

$$n^2 + n + \tfrac{1}{2} \geqslant n^2 - 1. \tag{24}$$

Finally, inequality (23) may be proved easily with the help of Hoelder's inequality.

In fact, Hoelder's inequality reads

$$\left| \sum_{n=1}^{N} a_n b_n \right| \leqslant \left(\sum_{n=1}^{N} |a_n|^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{N} |b_n|^q \right)^{\frac{1}{q}}, \tag{25}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{26}$$

Now we set $p = q = 2$ and $a_n = |c_n|$, $b_n = |c_{n-1}|$ and obtain

$$\sum_{n=1}^{N} |c_n||c_{n-1}| \leqslant \left(\sum_{n=1}^{N} |c_n|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} |c_{n-1}|^2 \right)^{\frac{1}{2}}. \tag{27}$$

Further we have

$$\left(\sum_{n=1}^{N} |c_{n-1}|^2 \right)^{\frac{1}{2}} = \left(\sum_{n=0}^{N-1} |c_n|^2 \right)^{\frac{1}{2}} \leqslant \left(\sum_{n=1}^{N} |c_n|^2 \right)^{\frac{1}{2}}, \tag{28}$$

where we used $c_0 = 0$. Inserting this last inequality into (27) just gives the inequality (23), which we wanted to prove.

The proof for the odd subspace (17) is completely analogous. Indeed, we find

$$\begin{aligned}\sin 2myH \sin 2ny = {} & \left(n^2 + \tfrac{1}{4}\right)[\cos 2(m-n)y - \cos(2(m+n)y]\\ & + \tfrac{1}{2}\left(n^2 + n + \tfrac{3}{4}\right)[\cos 2(m-n-1)y - \cos 2(m+n+1)y]\\ & + \tfrac{1}{2}\left(n^2 - n + \tfrac{3}{4}\right)[\cos 2(m-n+1)y - \cos 2(m+n-1)y]\end{aligned} \tag{29}$$

and, therefore,

$$\begin{aligned}\langle m|H|n\rangle &\equiv \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}y \sin 2myH \sin 2ny\\ &= \pi\left[\left(n^2 + \frac{1}{4}\right)\delta_{m,n} + \frac{1}{2}\left(n^2 + n + \frac{3}{4}\right)\delta_{m,n+1} + \frac{1}{2}\left(n^2 - n + \frac{3}{4}\right)\delta_{m,n-1}\right].\end{aligned} \tag{30}$$

For a general vector

$$|v\rangle = \sum_{n=0}^{N} c_n \sin 2ny, \tag{31}$$

we therefore obtain

$$\begin{aligned}\langle v|H|v\rangle &= \sum_{m,n=1}^{N} c_n \bar{c}_m \langle m|H|n\rangle\\ &= \pi \sum_{n=1}^{N}\left[c_n\bar{c}_n\left(n^2 + \frac{1}{4}\right) + c_n\bar{c}_{n-1}\frac{1}{2}\left(n^2 - n + \frac{3}{4}\right) + c_n\bar{c}_{n+1}\frac{1}{2}\left(n^2 + n + \frac{3}{4}\right)\right]\\ &\geqslant \pi \sum_{n=1}^{N}\left[c_n\bar{c}_n\left(n^2 + \frac{1}{4}\right) - |c_n||\bar{c}_{n-1}|\frac{1}{2}\left(n^2 - n + \frac{3}{4}\right) - |c_n||\bar{c}_{n+1}|\frac{1}{2}\left(n^2 + n + \frac{3}{4}\right)\right]\\ &= \pi \sum_{n=1}^{N}\left[|c_n|^2\left(n^2 + \frac{1}{4}\right) - |c_n||c_{n-1}|\left(n^2 - n + \frac{3}{4}\right)\right].\end{aligned} \tag{32}$$

Using the inequality

$$n^2 + \tfrac{1}{4} \geqslant n^2 - n + \tfrac{3}{4}, \tag{33}$$

the positive semi-definiteness of expression (32) is again implied by the inequality (23), which has been proved above.

Finally, let us mention that alternative stability proofs for the compact soliton of section 4.3 of [1], somewhat different in spirit than the one presented here, have been given recently in [3] (see section 6 of this reference) and [4].

References

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